Recall

Convergence $\Rightarrow$ Cesáro Summability $\Rightarrow$ Abel summability

$\not\Leftarrow$ $\qquad\qquad$ $\not\Leftarrow$

- (Tauberian Theorem)

  Abel summability + $c_n = o(\frac{1}{n})$ $\Rightarrow$ Convergence

- Cesáro summability + $c_n = O(\frac{1}{n})$ $\Rightarrow$ Convergence

  $\Downarrow$

  $n|c_n| < M$

Recall $S_N(f) = f * D_N$ where $D_N(\theta) = \sum_{n=-N}^{N} e^{in\theta} = \frac{\sin(N+\frac{1}{2})\theta}{\sin\frac{\theta}{2}}$

We have the convergence Thm for good kernels (Fejer kernel, Poisson kernel).

If K_n is a good kernel and f is continuous on $\mathbb{T}$, then $f * K_n \rightrightarrows f$ on $\mathbb{T}$.

But have $S_N(f)$ may not converge to f

even if f is continuous.

$\Rightarrow D_N$ is not a good kernel.

- $L_N := \frac{1}{2\pi}\int_{-\pi}^{\pi} |D_N(\theta)|\, d\theta \geq c \log N$

  $\Rightarrow L_N \to \infty$ as $N \to \infty$

(Tauberian Theorem)

If $\sum_{n=1}^{\infty} c_n$ is Abel summable to s and $n|c_n| \to 0$ as $n \to \infty$, then $\sum_{n=1}^{\infty} c_n = s$.

Pf: Fix $\varepsilon > 0$.

Since $\sum_{n=1}^{\infty} c_n$ is Abel summable to s, we have

$A(r) := \sum_{n=1}^{\infty} c_n r^n$ converges, $\forall r \in [0, 1)$.

$\lim_{r \to 1^-} A(r) = s$.

$$\begin{aligned}
\left| \sum_{n=1}^{N} c_n - s \right| &\leq \left| \sum_{n=1}^{N} c_n - A(r) \right| + \left| A(r) - s \right| \\
&\leq \left| \sum_{n=1}^{N} c_n - \sum_{n=1}^{N} c_n r^n \right| + \left| \sum_{n=N+1}^{\infty} c_n r^n \right| + \left| A(r) - s \right| \\
&\leq \left| (1-r) \sum_{n=1}^{N} c_n n \right| + \left| \sum_{n=N+1}^{\infty} c_n r^n \right| + \left| A(r) - s \right|
\end{aligned}$$

Note that

$1 - r^n = (1-r)(1 + \cdots + r^{n-1})$

$\leq n(1-r)$

Let $r = 1-\frac{1}{N}$

Take $N_0 \in \mathbb{N}$ s.t. $N > N_0$ we have

$\frac{1}{N}\sum_{n=1}^{N} n|c_n| < \varepsilon$

$n|c_n| < \varepsilon$

$|A(1-\frac{1}{N})-s| < \varepsilon$

$$\le \left|\frac{1}{N}\sum_{n=1}^{N} c_n n\right| + \left|\sum_{n=N+1}^{\infty} c_n\left(1-\frac{1}{N}\right)^n\right| + \left|A\left(1-\frac{1}{N}\right)-s\right|$$

$$\le \frac{1}{N}\sum_{n=1}^{N} n|c_n| + \frac{1}{N}\sum_{n=N+1}^{\infty}|c_n|\, n\left(1-\frac{1}{N}\right)^n + \left|A\left(1-\frac{1}{N}\right)-s\right|$$

$$\le 2\varepsilon + \varepsilon\left(\frac{1}{N}\sum_{n=N+1}^{\infty}\left(1-\frac{1}{N}\right)^n\right)$$

$$= 2\varepsilon + \varepsilon\frac{1}{N}\cdot\frac{\left(1-\frac{1}{N}\right)^{N+1}}{1-\left(1-\frac{1}{N}\right)}$$

$$= 2\varepsilon + \varepsilon\left(1-\frac{1}{N}\right)^{N+1} \to \frac{1}{e}$$

$$< 2\varepsilon + M\varepsilon.$$

$\square$

- If $\sum_{n=1}^{\infty} c_n$ is Cesáro summable to s and $n|c_n| < M$, then $\sum_{n=1}^{\infty} c_n = s$.

$\sigma_N \to s$ as N

Pf: $S_N = \sum_{n=1}^{N} c_n$, $\sigma_N = \frac{1}{N}\sum_{n=1}^{N} s_n = \frac{1}{N}\sum_{n=1}^{N}\sum_{k=1}^{n} c_k = \frac{1}{N}\sum_{n=1}^{N}(N-n+1)c_n$

Fix $\varepsilon > 0$,

$|s_N - s| \le |s_N - \sigma_N| + |\sigma_N - s|$

We want $|s_N - \sigma_N| \to 0$ as $N \to \infty$.

$$|s_N - \sigma_N| = \left|\frac{1}{N}\sum_{n=1}^{N} N c_n - \frac{1}{N}\sum_{n=1}^{N}(N-n+1)c_n\right|$$

$$= \frac{1}{N}\left|\sum_{n=1}^{N}(n-1)c_n\right|$$

$$\le \frac{1}{N}\sum_{n=1}^{N}(n-1)|c_n| < \frac{1}{N}\sum_{n=1}^{N} M = M$$

For $m > n$

$$m\sigma_m - n\sigma_n = \sum_{k=1}^{m}(m-k+1)c_k - \sum_{k=1}^{n}(n-k+1)c_k$$

$$= \sum_{k=1}^{n}(m-n)c_k + \sum_{k=n+1}^{m}(m-k+1)c_k$$

$$= (m-n)s_n + \sum_{k=n}^{m-1}(m-k)c_{k+1}$$

$$(m-n)s_n = m\sigma_m - n\sigma_n - \sum_{k=n}^{m-1}(m-k)c_{k+1}$$

$$(m-n)(s_n-\sigma_n) = m(\sigma_m-\sigma_n) - \sum_{k=n}^{m-1}(m-k)c_{k+1}$$

$$s_n-\sigma_n = \frac{m}{m-n}(\sigma_m-\sigma_n) - \frac{m}{m-n}\sum_{k=n}^{m-1}\left(1-\frac{k}{m}\right)c_{k+1}$$

$$|s_n-\sigma_n| \le \frac{m}{m-n}|\sigma_m-\sigma_n| + \frac{m}{m-n}\sum_{k=n}^{m-1}\left(\frac{1}{k}-\frac{1}{m}\right)k|c_{k+1}|$$

$$\le \frac{1+2\varepsilon}{\varepsilon}\cdot\varepsilon^2 + M\left(\frac{m}{m-n}\sum_{k=n}^{m-1}\frac{1}{k} - \overset{1}{\cancel{\frac{m}{m-n}\cdot\frac{m-n}{m}}}\right)$$

$$\le (1+2\varepsilon)\varepsilon + M\left(\frac{1+2\varepsilon}{\varepsilon}\log\left(\frac{m-1}{n-1}\right) - 1\right)$$

$$\le (1+2\varepsilon)\varepsilon + M\left(\frac{1+2\varepsilon}{\varepsilon}\log\left(1+\frac{[n\varepsilon]}{n-1}\right) - 1\right)$$

Choose $m = n + [\varepsilon n]$

We have $\frac{m}{m-n} \to \frac{\varepsilon+1}{\varepsilon}$ as $n\to\infty$

Take $N_0 \in \mathbb{N}$ s.t. $\forall\ n > N_0$

$$\frac{m}{m-n} < \frac{1+2\varepsilon}{\varepsilon}$$

$$|\sigma_m - \sigma_n| < \varepsilon^2$$

$$\frac{n}{n-1} < 1+\varepsilon$$

$\log(1+x) \le x$

$\frac{n}{n-1} \to 1$ as $n \to \infty$

$$\le (1+2\varepsilon)\varepsilon + M\left(\frac{1+2\varepsilon}{\varepsilon}\frac{[n\varepsilon]}{n-1} - 1\right)$$

$$\le (1+2\varepsilon)\varepsilon + M\left(\frac{\cancel{1}+2\varepsilon}{\cancel{\varepsilon}}\cancel{\varepsilon}(\cancel{1}+\varepsilon) - \cancel{1}\right)$$

$$\le (1+2\varepsilon)\varepsilon + M(2\varepsilon^2+3\varepsilon).$$

$\square$

- Prove $L_N := \frac{1}{2\pi}\int_{-\pi}^{\pi} |D_N(\theta)|\, d\theta = \frac{1}{2\pi}\int_{-\pi}^{\pi}\left|\frac{\sin(N+\frac{1}{2})\theta}{\sin\frac{\theta}{2}}\right| d\theta > C\log N$

Pf: Obeserve $|\sin x| < |x|$, $\forall x \in \mathbb{R}$ and the integrand is even, it suffices to show

$$\int_0^{\pi}\left|\frac{\sin(N+\frac{1}{2})\theta}{\theta}\right| d\theta > C\log N$$

Let $x = (N+\frac{1}{2})\theta$, then $dx = (N+\frac{1}{2})\,d\theta$

$$\int_0^{\pi}\left|\frac{\sin(N+\frac{1}{2})\theta}{\theta}\right| d\theta$$

$$= \int_0^{(N+\frac{1}{2})\pi} \frac{|\sin x|}{\frac{x}{\cancel{N+\frac{1}{2}}}}\, \frac{dx}{\cancel{N+\frac{1}{2}}}$$

$$= \int_0^{(N+\frac{1}{2})\pi} \frac{|\sin x|}{x}\,dx$$

$$= \sum_{n=1}^{N} \int_0^{\pi} \frac{|\sin((n-1)\pi + y)|}{(n-1)\pi + y}\,dy$$

$|\sin x|$ is π-periodic

$$= \sum_{n=1}^{N} \int_0^{\pi} \frac{\sin y}{(n-1)\pi + y}\,dy$$

$$\geq \sum_{n=1}^{N} \frac{1}{n\pi} \int_0^{\pi} \sin y\,dy$$

$$= \frac{2}{\pi} \sum_{n=1}^{N} \frac{1}{n}$$

$$\geq \frac{2}{\pi} \sum_{n=1}^{N} \int_n^{n+1} \frac{1}{x}\,dx$$

$$= \frac{2}{\pi} \int_1^{N+1} \frac{1}{x}\,dx$$

$$= \frac{2}{\pi} \log(N+1) \geq \frac{2}{2} \log N.$$

$\square$